Neighborhoods of S^1-like Continua in 4-Manifolds

FREDRIC D. ANCEL, VO THANH LIEM,[1] & GERARD A. VENEMA[2]

0. Introduction

In this paper we study the problem of determining which compact subsets of 4-manifolds have close neighborhoods that collapse to 1-dimensional spines. As is explained in [10], the study of this problem is motivated by the desire to understand engulfing of 2-dimensional polyhedra in piecewise linear 4-manifolds. The technology of 4-manifold topology does not seem to be well enough developed for us to characterize such compacta completely. We restrict our attention, therefore, to the case in which the neighborhood collapses to a copy of the circle, S^1. In that case the fundamental groups which arise are infinite cyclic, so that we can apply the $\mathbf{Z}$-theory of Freedman and Quinn [2; 3]. Our main theorem characterizes those compact subsets of 4-manifolds that have arbitrarily close neighborhoods with spines homeomorphic to S^1.

THEOREM 1. *Suppose X is a compact subset of the orientable 4-manifold M^4. Then X has arbitrarily close neighborhoods homeomorphic to $S^1 \times B^3$ if and only if*

(1) *X has the shape of some S^1-like continuum, and*
(2) *X satisfies the inessential loops condition.*

Let Y be an S^1-like continuum. Then Y is the inverse limit of an inverse sequence in which each space is S^1. Thus there is a standard embedding of Y in S^4 as the intersection of a nested sequence of thin tubes, each tube homeomorphic with $S^1 \times B^3$. We will identify Y with this embedded copy of Y. The following complement theorem is then a corollary to Theorem 1. We use $\mathrm{Fd}(X)$ to denote the fundamental dimension of X.

COROLLARY. *Suppose X is a compact subset of S^4, $\mathrm{Fd}(X)=1$, X satisfies the inessential loops condition, and Y is an S^1-like continuum standardly embedded in S^4. Then $S^4 - X \cong S^4 - Y$ if and only if $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.*

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A less general theorem than Theorem 1 is proved in [10]. The theorem proved in [10] gives sufficient conditions for the existence of arbitrarily close neighborhoods homeomorphic to $S^1 \times B^3$, but the conditions are not necessary. There are some crucial differences between the proof given here and that in [10]. But the main construction of the present proof is based on that in [10], so we will have occasion to refer to [10] in the course of proving Theorem 1. The reader will need to consult [10] to find some of the details of the proofs in this paper.

1. Definitions and Notation

Suppose X is a compact subset of the interior of the n-manifold M. We say that X satisfies the *inessential loops condition* (ILC) if for every neighborhood U of X in M there exists a neighborhood V of X in U such that each loop in $V-X$ which is homotopically inessential in V is also inessential in $U-X$. We use "$\simeq$" to denote "is homotopic to" and "$\cong$" to denote either "is homeomorphic to" or "is isomorphic to", depending on the context. When we say that X has *arbitrarily close neighborhoods homeomorphic to* $S^1 \times B^3$, we mean that for every neighborhood U of X there exists a neighborhood N of X such that $N \subset U$ and $N \cong S^1 \times B^3$.

The statement $\text{Sh}(X) = \text{Sh}(Y)$ means that X and Y have the same shape. The *fundamental dimension* of X is defined by $\text{Fd}(X) = \min\{\dim Y: \text{Sh}(X) = \text{Sh}(Y)\}$. Let P be a polyhedron. A space X is said to be *P-like* if X can be written as the inverse limit of an inverse sequence in which each space is homeomorphic to P. This is equivalent to the assertion that for every $\epsilon > 0$, there exists an onto map $f: X \to P$ such that the diameter of $f^{-1}(y)$ is smaller than ϵ for every $y \in P$. An S^1-like continuum is also called a *solenoid*.

Consult [4] for other definitions related to shape theory. It is not necessary to be familiar with very much shape theory in order to read this paper. In fact, the main use of shape theory is in the following characterization of compacta which have the shape of S^1-like continua. The proposition follows easily from the definitions in shape theory (cf. [4]).

PROPOSITION 1.1. *Let X be a compact subset of the n-manifold M. Then X has the shape of some S^1-like continuum if and only if, for every neighborhood U of X, there exists a smaller neighborhood V of X in U and maps $\alpha: V \to S^1$ and $\beta: S^1 \to U$ such that $\beta \circ \alpha \simeq$ inclusion.*

If P is a polyhedron endowed with a triangulation and i is an integer, then $P^{(i)}$ denotes the i-skeleton of P in that triangulation. We use the notation $P \searrow L$ to mean that P collapses to L.

2. Beginning of the Proof of Theorem 1

This section contains the proof of one direction of Theorem 1 and the proof of the corollary. The remainder of the paper is devoted to a proof of the converse direction of Theorem 1.

Suppose that X has arbitrarily close neighborhoods in M that are homeomorphic to $S^1 \times B^3$. Let a neighborhood U of X be given and let

$$f: (B^2, \partial B^2) \to (U, U-X)$$

be a map. There exists a neighborhood N of X such that $N \subset U - f(\partial B^2)$ and $N \cong S^1 \times B^3$. The neighborhood N has a PL structure induced by the homeomorphism $N \cong S^1 \times B^3$. Approximate $f \mid f^{-1}(\operatorname{int} N)$ by a map which is in general position with respect to that PL structure. This gives a new continuous function $f': B^2 \to U$ which agrees with f on ∂B^2 but which also has the property that $f'(B^2)$ misses the core, $S^1 \times \{0\}$, of N. Use the radial structure of $B^3 - \{0\}$ to push the image of f' out of $\operatorname{int} N$ and hence off X. This shows that X satisfies ILC. Since $X = \bigcap_{i \geq 1} N_i$, where each $N_i \cong S^1 \times B^3$ and $N_{i+1} \subset \operatorname{int} N_i$, we see that X is homeomorphic to the inverse limit of an inverse sequence in which each space has the homotopy type of S^1. It follows that X has the shape of an S^1-like continuum. This proves that the conditions (1) and (2) listed in Theorem 1 are necessary.

Having completed the proof of one direction of Theorem 1, we now turn to the proof of the corollary. Suppose X and Y are as in the statement of the corollary and that $S^4 - X \cong S^4 - Y$. Then $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$ by [11]. Conversely, if $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$, then $S^4 - X \cong S^4 - Y$ by [10, Thm. 4.3].

3. Constructing Neighborhoods

Let X be a compact subset of the interior of an orientable 4-manifold M, and assume that X satisfies conditions (1) and (2) of Theorem 1. Since we work within $\operatorname{int} M$, we may assume $\partial M = \emptyset$. We may also assume that M is a piecewise linear manifold because every connected 4-manifold has a PL structure in the complement of a point [6, Cor. 2.2.3]. Let U be a neighborhood of X in M. Our goal is to find a neighborhood N of X in U which is homeomorphic to $S^1 \times B^3$. In this section we describe the construction of a special sequence of neighborhoods of X which will be used to prove the existence of N.

Let U_0 be a compact connected PL manifold neighborhood of X in U. Because X has the shape of an S^1-like continuum, there is a neighborhood U_1 of X and a polyhedron K_1 in U_0, $K_1 \cong S^1$, such that the inclusion map of U_1 in U_0 is homotopic in U_0 to a map $\beta_1: U_1 \to K_1$. (See Proposition 1.1.) By [10, Lemma 1.1], we can push K_1 off X. So by replacing U_1 by a smaller neighborhood, we can arrange that $U_1 \cap K_1 = \emptyset$. We may also assume that U_1 is a compact, connected PL manifold. By [9, Thm. 3.1], X does not separate U_1. Thus we can find a finite collection of PL arcs in $U_1 - X$ which connect all the components of ∂U_1. We remove a small regular neighborhood of the union of these arcs from U_1; the result is a new U_1 with the additional property that $U_0 - U_1$ is connected. This construction is continued inductively to define a sequence of neighborhoods $U_0, U_1, \ldots$ and a sequence of 1-dimensional polyhedra $K_1, K_2, \ldots$ which satisfy the following properties.

(1) U_i is a compact connected PL manifold neighborhood of X in the interior of U_{i-1}.
(2) U_i does not separate U_{i-1}.
(3) K_i is a compact polyhedron in $U_{i-1}-U_i$ such that $K_i \cong S^1$.
(4) There is a homotopy $f_i: U_i \times [0,1] \to U_{i-1}$ such that $f_i(x,0)=x$ and $f_i(x,1) \in K_i$ for every $x \in U_i$.

(Below we will modify f_i, when i is odd in a way that weakens this statement, to the assertion that the image of f_i lies in U_{i-2}.)

Let $\hat{f}_i: K_{i+1} \to K_i$ be the map defined by $\hat{f}_i(x)=f_i(x,1)$. Since we have assumed that X has the shape of a nontrivial S^1-like continuum, we may add that

(5) $\hat{f}_i: K_{i+1} \to K_i$ is essential for every i.

So far we have used only the fact that X has the shape of an S^1-like continuum. The ILC hypothesis allows us to gain some control over π_1 as well. By [8], $\pi_1(U_i, U_i - X) = 0 = \pi_2(U_i, U_i - X)$. So we can push $f_i(K_{i+1} \times [0,1])$ off X. We do so and then inductively choose U_{i+1} small enough so that $U_{i+1} \cap f_i(K_{i+1} \times [0,1]) = \emptyset$.

Put each of the maps $f_i | K_{i+1} \times [0,1]$ in general position, keeping $f_i | K_{i+1} \times \{0,1\}$ fixed. Then

$$f_i(K_{i+1} \times [0,1)) \cap f_{i+1}(K_{i+2} \times [0,1])$$

will consist of a finite number of points. If $i \geq 2$ is even then take a small neighborhood in $f_{i+1}(K_{i+2} \times [0,1])$ of each such intersection point and push the neighborhood along $f_i(K_{i+1} \times [0,1])$ until it is pushed off the $f_i(K_{i+1} \times \{0\})$-end of $f_i(K_{i+1} \times [0,1])$. This removes the points of intersection between $f_i(K_{i+1} \times [0,1))$ and $f_{i+1}(K_{i+2} \times [0,1])$. The price we must pay is that there are new self-intersections introduced in $f_{i+1}(K_{i+2} \times [0,1])$, and $f_{i+1}(K_{i+2} \times [0,1])$ is stretched out so that it no longer stays in U_i, but now maps into U_{i-1}. We can therefore add the following four additional conditions to the list of properties satisfied by the sequences constructed thus far.

(6) $f_i(K_{i+1} \times [0,1]) \cap f_{i+1}(K_{i+2} \times [0,1]) = K_{i+1}$ if $i \geq 2$ is even.
(7) $f_i(K_{i+1} \times [0,1]) \cap f_j(K_{j+1} \times [0,1]) = \emptyset$ for $|i-j|>1$.
(8) $f_i(K_{i+1} \times [0,1]) \subset U_{i-1} - U_{i+1}$ if i is even.
(9) $f_i(K_{i+1} \times [0,1]) \subset U_{i-2} - U_{i+1}$ if i is odd.

We combine $f_{2i-1} | K_{2i} \times [0,1]$ and $f_{2i} | K_{2i+1} \times [0,1]$ into a single homotopy g_i by first running $f_{2i} | K_{2i+1} \times [0,1]$ at double speed and then running

$$f_{2i-1} \circ ((f_{2i,1} | K_{2i+1}) \times \mathrm{id}_{[0,1]})$$

at double speed. Set $T_i = g_i(K_{2i+1} \times [0,1))$ for $i \geq 1$. Then

(10) $g_i: K_{2i+1} \times [0,1] \to (\operatorname{int} U_{2i-3}) - U_{2i+1}$ is a homotopy such that $g_{i,0} = \mathrm{id}$ on K_{2i+1}, $g_i(K_{2i+1} \times \{1\}) = K_{2i-1}$, and $T_i \cap T_j = \emptyset$ for $i \neq j$.

(In order that this and later statements make sense when $i=1$, we set $U_{-1} = U_0$.)

LEMMA 3.1. *If X satisfies the inessential loops condition, then the neighborhoods $U_1, U_2, U_3, \dots$ can be chosen so that the inclusion-induced homomorphism $\pi_2(U_i, U_i - U_j) \to \pi_2(U_{i-1}, U_{i-1} - U_{j+1})$ is the trivial homomorphism whenever $1 \leq i < j$.*

Proof. The proof is the same as that of [10, Lemma 2.1]. □

We now construct a new sequence $V_0, V_1, V_2, \dots$ of neighborhoods of X. These new neighborhoods will improve on the U_i's in the following sense: V_i will contain a copy K_i' of S^1 such that for each $i \geq 2$, there is a homotopy of V_i in V_{i-2} to K_i' which keeps K_i' fixed. Thus V_i will homotopically mimic a regular neighborhood of K_i'.

Begin by letting $V_0 = U_0$, $K_1' = K_1$, N_1 be a regular neighborhood of K_1' in V_0, and $h_1 = \mathrm{id}_M$. Approximate g_1 with a general position map of $K_3 \times [0,1]$. Now $g_1(K_3 \times \{1\})$ is no longer a subset of K_1, but is still contained in N_1 and is homotopic to the original there. Since g_1 is in general position, the only singularities will be a finite number of double points. Let g_1' be the embedding of $K_3 \times [0,1]$ obtained by piping each of these double points off the $(K_3 \times \{1\})$-end of $g_1(K_3 \times [0,1])$. We define L_1 to be $g_1'(K_3 \times [0,1])$. Now choose two relative regular neighborhoods P_1 and P_1' of L_1 modulo K_3 in such a way that P_1' is much thinner than P_1 is. We want these two neighborhoods to fit together correctly near K_3. The simplest way to accomplish this is to be specific about their construction: Start with a triangulation of U_0 which includes L_1 as a subcomplex and then define P_1 to be the union of all simplices in the second barycentric subdivision which meet $L_1 - K_3$, and define P_1' to be the union of all simplices in the fourth barycentric subdivision which lie in P_1 and meet L_1. We then define U_1' to be $\mathrm{cl}(U_1 - P_1)$ and define V_1 to be $U_1' \cup P_1'$. The construction of V_1 is illustrated schematically in Figure 1. It is important to notice that condition (6) implies that $g_2(K_5 \times [0,1]) \cup U_3 \subset V_1$.

The construction of V_2 is similar to that of V_1. Begin by setting $K_2' = g_1'(K_3 \times \{1\})$ and by choosing a regular neighborhood N_2 of K_2' in $N_1 \cap P_1'$. Now L_1 is homeomorphic to $K_3 \times [0,1]$, and shrinking out its fibers defines a map $L_1 \to g_1'(K_3 \times \{1\})$ which can be approximated by a homeomorphism h_2 of M such that h_2 is the identity off a close neighborhood of L_1. Put $g_2: K_5 \times [0,1] \to U_1'$ in general position and pipe the singularities over the $(K_5 \times \{1\})$-end to get an embedding g_2' of $K_5 \times [0,1]$ with the property that $g_2'(K_5 \times \{1\}) \subset h_2^{-1}(N_2)$. We let $L_2' = g_2'(K_5 \times [0,1])$ and define L_2 to be $h_2(L_2')$. Notice that L_2 is just L_2' stretched out so that it stretches all the way from K_5 into N_2. We define K_3' to be the end of L_2 which is in N_2; that is, $K_3' = h_2(g_2'(K_5 \times \{1\}))$. Let P_2 and P_2' be a pair of relative regular neighborhoods of L_2 modulo K_5 (defined in a way which is analogous to the definition of P_1 and P_1' above), and define $U_3' = \mathrm{cl}(U_3 - P_2)$ and $V_2 = U_3' \cup P_2'$.

We continue this construction inductively, generating sequences $\{K_i'\}$, $\{N_i\}$, $\{h_i\}$, $\{g_i'\}$, $\{L_i\}$, $\{P_i\}$, $\{P_i'\}$, $\{U_{2i-1}'\}$, and $\{V_i\}$ satisfying the following conditions for $i \geq 2$.

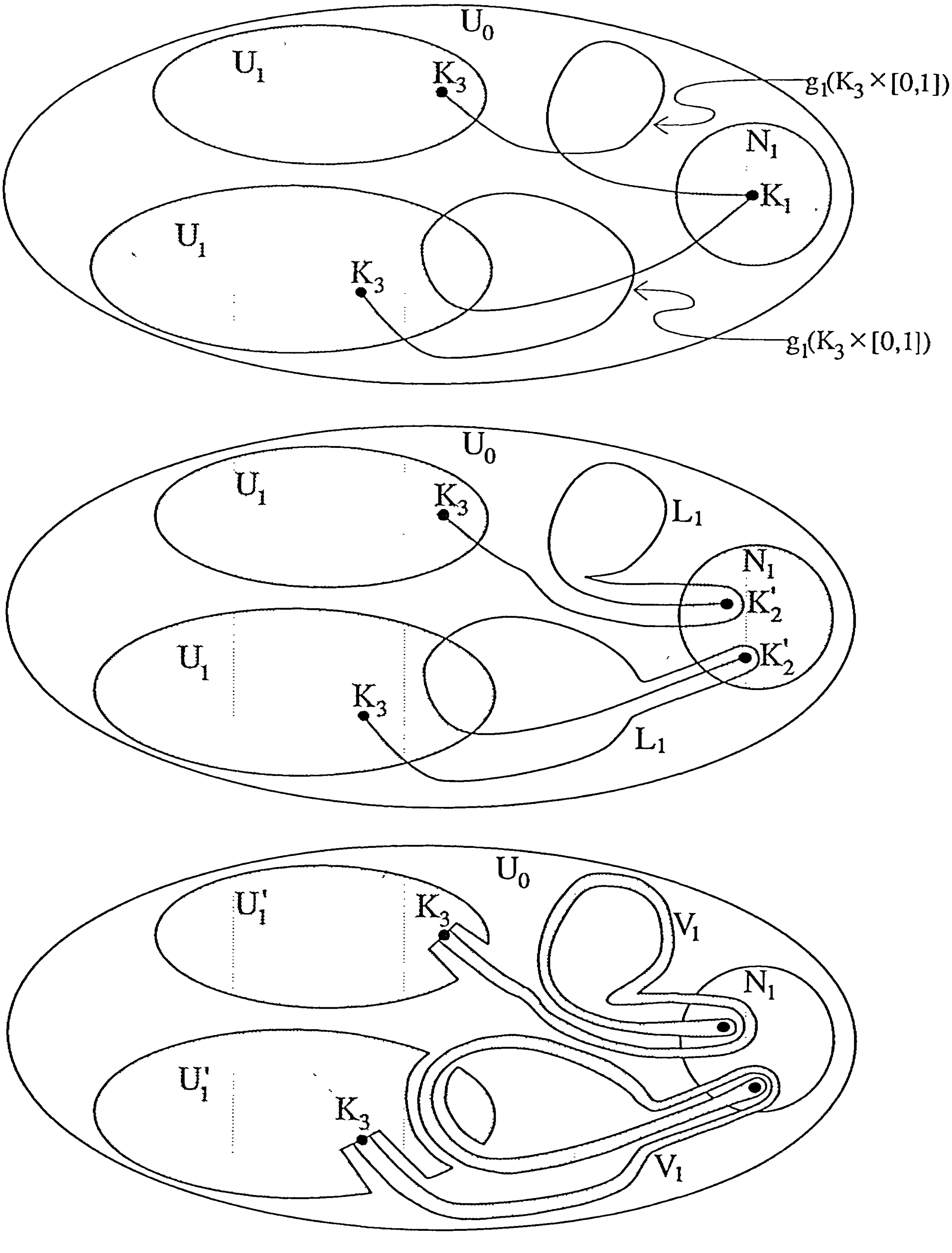

Figure 1

(11) $g_i(K_{2i+1}\times[0,1])\subset(\operatorname{int}(V_{i-1}\cap U_{2i-3}))-(U_{2i+1}\cup T_{i+1})$.
(12) K_i' is the polyhedral 1-sphere $h_{i-1}\circ g_{i-1}'(K_{2i-1}\times\{1\})$.
(13) N_i is a regular neighborhood of K_i' in the interior of $N_{i-1}\cap V_{i-1}$.
(14) h_i is a piecewise linear homeomorphism of M such that $h_i(L_{i-1})\subset\operatorname{int}N_i$ and $h_i=\mathrm{id}$ on K_i' and outside $V_{i-1}-U_{2i-1}$.

(L_{i-1} is a polyhedral annulus in $((\operatorname{int}V_{i-1})-U_{2i-1})$ with boundary components K_{2i-1} and K_i', and $K_i'\subset\operatorname{int}N_i$. h_i squeezes L_{i-1} up its fibers into N_i.

Hence, $g_i(K_{2i+1}\times\{1\})=K_{2i-1}\subset L_{i-1}\subset h_i^{-1}(\operatorname{int} N_i)$. Also, $h_i=\mathrm{id}$ on K_{2i+1} because $K_{2i+1}\subset U_{2i-1}$.)

(15) $g_i': K_{2i+1}\times[0,1]\to[(\operatorname{int}(V_{i-1}\cap U_{2i-3}))-(U_{2i+1}\cup T_{i+1})]$ is a PL embedding which is homotopic to g_i in $(\operatorname{int}(V_{i-1}\cap U_{2i-3}))-(U_{2i+1}\cup T_{i+1})$ such that $g_i'(K_{2i+1}\times\{0\})=K_{2i+1}$ and $g_i'(K_{2i+1}\times\{1\})\subset h_i^{-1}(\operatorname{int} N_i)$. $g_i'\mid K_{2i+1}\times\{1\}$ is homotopic to $g_i\mid K_{2i+1}\times\{1\}$ in $h_i^{-1}(\operatorname{int} N_i)$.

(g_i' is obtained from g_i by putting g_i in general position and then piping its double points off the $(K_{2i+1}\times\{1\})$-end.)

(16) $L_i=h_i\circ g_i'(K_{2i+1}\times[0,1])$ is a polyhedral annulus in $(\operatorname{int} V_{i-1})-(U_{2i+1}\cup T_{i+1})$ with boundary components K_{2i+1} and $K_{i+1}'=h_i\circ g_i'(K_{2i+1}\times\{1\})\subset\operatorname{int} N_i$.

(17) P_i and P_i' are relative regular neighborhoods of L_i modulo K_{2i+1} in $(\operatorname{int} V_{i-1})-(U_{2i+1}\cup T_{i+1})$ such that $P_i'\subset P_i$ and $P_i'\cap\partial P_i$ is a regular neighborhood of K_{2i+1} in $(\partial P_i)\cap(\operatorname{int} U_{2i-1})$.

(18) $U_{2i-1}'=\mathrm{cl}(U_{2i-1}-P_i)$ is a PL 4-manifold such that $U_{2i-1}'\cap L_i=K_{2i+1}$, $U_{2i+1}\subset\operatorname{int} U_{2i-1}'$, and $g_{i+1}(K_{2i+3}\times[0,1])\subset(\operatorname{int} U_{2i-1}')\cup K_{2i+1}$.

(19) $V_i=U_{2i-1}'\cup P_i'$ is a PL 4-manifold such that $V_i\searrow U_{2i-1}'\cup L_i$, $U_{2i+1}\cup g_{i+1}(K_{2i+3}\times[0,1])\subset\operatorname{int} V_i$, and $V_i\subset\operatorname{int} V_{i-1}$.

4. Homotopy Properties of Neighborhoods

The most important properties of the neighborhoods which were constructed in the previous section are spelled out in the lemmas of this section.

LEMMA 4.1. *Whenever $i\geq 2$, the inclusion map of V_i in V_{i-2} is homotopic in V_{i-2} to a map $\rho: V_i\to N_i$ via a homotopy which keeps N_{i+1} fixed.*

Proof. Because of property (19), it suffices to define a homotopy of $U_{2i-1}'\cup L_i$ in V_{i-2} which squeezes $U_{2i-1}'\cup L_i$ into N_i and which is stationary on $L_i\cap N_{i+1}$. Define the homotopy α of U_{2i-1}' by $\alpha=h_i\circ f_{2i-1}\mid U_{2i-1}'\times[0,1]$. $\alpha(U_{2i-1}'\times[0,1])\subset V_{i-2}$ because $f_{2i-1}(U_{2i-1}\times[0,1])\subset U_{2i-3}\subset V_{i-2}$ and h_i is supported on $V_{i-1}\subset V_{i-2}$. $\alpha_0=h_i\mid U_{2i-1}'=\mathrm{id}\mid U_{2i-1}'$, and $\alpha_1(U_{2i-1}')\subset h_i(K_{2i-1})\subset N_i$. There is a homotopy $\beta: L_i\times[0,1]\to V_i$ which squeezes the annulus L_i into N_i, and which is stationary on $L_i\cap N_{i+1}$ and satisfies the condition that $\beta\mid K_{2i+1}\times[0,1]=h_i\circ g_i'$. We would like to define a homotopy of $U_{2i-1}'\cup L_i$ by taking the union of α and β. This doesn't work because α and β disagree on the overlap $U_{2i-1}'\cap L_i=K_{2i+1}$. However, as we shall now argue, $\alpha\mid K_{2i+1}\times[0,1]$ and $\beta\mid K_{2i+1}\times[0,1]$ are homotopic. This allows us to deform β to a homotopy which agrees with α on $K_{2i+1}\times[0,1]$.

We define a homotopy $\Phi: (K_{2i+1}\times[0,1])\times[0,1]\to(U_{2i-3}-U_{2i+1})$ from $f_{2i-1}\mid K_{2i+1}\times[0,1]$ to g_i by

$$\Phi((x,s),t)=\begin{cases} f_{2i}(x,2s) & \text{for } ((x,s),t)\in(K_{2i+1}\times[0,t/2])\times[0,1],\\ f_{2i-1}(f_{2i}(x,t),(2s-t)/(2-t)) & \text{for } ((x,s),t)\in(K_{2i+1}\times[t/2,1])\times[0,1].\end{cases}$$

Then Φ is stationary on $K_{2i+1}\times\{0\}$ and keeps the image of $K_{2i+1}\times\{1\}$ in $h_{i-1}^{-1}(N_i)$. So by (15) there is homotopy from $f_{2i-1}|K_{2i+1}\times[0,1]$ to g_i' in $U_{2i-3}-U_{2i+1}$ which is stationary on $K_{2i+1}\times\{0\}$ and keeps the image of $K_{2i+1}\times\{1\}$ in $h_{i-1}^{-1}(N_i)$. By composing this homotopy with h_i, we obtain a homotopy from $\alpha|K_{2i+1}\times[0,1]$ to $\beta|K_{2i+1}\times[0,1]$ in V_{i-2} which is stationary on $K_{2i+1}\times\{0\}$ and keeps the image of $K_{2i+1}\times\{1\}$ in N_i.

Now, using the fact that $K_{2i+1}\times[0,1]$ is collared in $L_i\times[0,1]$ (or the fact that M is an ANR), we can deform β to a homotopy $\gamma: L_i\times[0,1]\to V_{i-2}$ which squeezes the annulus L_i into N_i, which is stationary on $L_i\cap N_{i+1}$, and which agrees with α on $K_{2i+1}\times[0,1]$. Now $\alpha\cup\gamma$ is a homotopy of $U_{2i-1}'\cup L_i$ in V_{i-2} which squeezes $U_{2i-1}'\cup L_i$ into N_i and which is stationary on $L_i\cap N_{i+1}$. □

LEMMA 4.2. *The inclusion-induced homomorphism*

$$\pi_2(V_i, V_i-V_j)\to\pi_2(V_i, V_i-V_{j+2})$$

is trivial whenever $1\le i<j$.

Proof. Let

$$g:(B^2,\partial B^2)\to(V_i, V_i-V_j)$$

represent an element of $\pi_2(V_i, V_i-V_j)$. Observe that $g(\partial B^2)\cap U_{2j+1}=\emptyset$ because $U_{2j+1}\subset V_j$. Let τ be a triangulation of B^2 so fine that the g-image of any simplex of τ which intersects U_{2j+1} lies in U_{2j}. We now use the facts that (i) $U_{2j}-U_{2j+1}$ is connected, (ii) $\pi_1(U_{2j}, U_{2j}-U_{2j+1})=0$ by [10, Lemma 1.1], and (iii) $U_{2j}\subset U_{2i+2}\subset V_i$, to homotope $g\operatorname{rel}\partial B^2$ in V_i so that g maps the 1-skeleton of τ into V_i-U_{2j+1}, while retaining the condition that the g-image of any simplex of τ which intersects U_{2j+1} lies in U_{2j}. Lemma 3.1 tells us that $\pi_2(U_{2j}, U_{2j}-U_{2j+1})\to\pi_2(U_{2j-1}, U_{2j-1}-U_{2j+2})$ is the zero homomorphism. Also $U_{2j-1}\subset U_{2i+1}\subset V_i$. Using these observations and moving g only on the interior of those 2-simplices of τ whose g-image intersects U_{2j+1}, we homotope $g\operatorname{rel}\partial B^2$ in V_i so that $g(B^2)\cap U_{2j+2}=\emptyset$. Hence, $g(B^2)\cap U_{2j+3}'=\emptyset$. We then pipe the intersections of $g(B^2)$ and L_{j+2} off the K_{j+3}'-end of L_{j+2}. So $g(B^2)\cap(U_{2j+3}'\cup L_{j+2})=\emptyset$. Finally, using the collapse $V_{j+2}\searrow U_{2j+3}'\cup L_{j+2}$, we push $g(B^2)$ off V_{j+2}. Thus, we have homotoped $g\operatorname{rel}\partial B^2$ in V_i so that $g(B^2)\subset V_i-V_{j+2}$. □

The following lemma is a simple version of the eventual Hurewicz lemma of [5].

LEMMA 4.3. *Let $k\ge 2$. For $-2\le r\le k$, let $B_r\subset A_r$ be polyhedra such that A_r is path-connected, and let $\varphi_r:(A_{r-1},B_{r-1})\to(A_r,B_r)$ be a map. Suppose*

(a) $(\varphi_{-1})_*: H_k(A_{-2},B_{-2})\to H_k(A_{-1},B_{-1})$ *is zero,*
(b) $(\varphi_r)_\#: \pi_r(A_{r-1},B_{r-1})\to\pi_r(A_r,B_r)$ *is zero for* $0\le r\le k-1$, *and*
(c) $(\varphi_k)_\#: \pi_1(B_{k-1})\to\pi_1(B_k)$ *is zero.*

Then $(\varphi_k\circ\cdots\circ\varphi_0\circ\varphi_{-1})_\#: \pi_k(A_{-2},B_{-2})\to\pi_k(A_k,B_k)$ *is zero.*

Proof. We follow the notation of [7]: $\pi_r'(X, Y)$ denotes the quotient of $\pi_r(X, Y)$ by the normal subgroup generated by elements of the form

$$(*) \qquad (h_{[\omega]}[\alpha])[\alpha]^{-1},$$

where $[\alpha] \in \pi_r(X, Y)$, ω is a loop in Y, and $h_{[\omega]}[\alpha]$ denotes $[\alpha]$ acted on by $[\omega]$ under the usual action of the fundamental group on $\pi_r(X, Y)$. The Hurewicz isomorphism theorem [7, Thm. 7.5.4] says that if X and Y are path-connected and (X, Y) is $(r-1)$-connected, then the Hurewicz homomorphism $\pi_r'(X, Y) \to H_r(X, Y)$ is an isomorphism.

Triangulate (A_{-1}, B_{-1}) and let $A_{-1}^{(k-1)}$ denote the $(k-1)$-skeleton of A_{-1}. Consider the following commutative rectangle:

$$\begin{array}{ccc} \pi_k'(A_{-2}, B_{-2}) & \xrightarrow{(\varphi_{-1})_\#} & \pi_k'(A_{-1}, A_{-1}^{(k-1)} \cup B_{-1}) \\ \eta \downarrow & & \downarrow \eta' \\ H_k(A_{-2}, B_{-2}) & \xrightarrow[(\varphi_{-1})_*]{} & H_k(A_{-1}, A_{-1}^{(k-1)} \cup B_{-1}). \end{array}$$

In this rectangle, $(\varphi_{-1})_* = 0$ because it factors through the zero homomorphism of hypothesis (a). η and η' are Hurewicz homomorphisms. Since A_{-1} and $A_{-1}^{(k-1)} \cup B_{-1}$ are path-connected and $(A_{-1}, A_{-1}^{(k-1)} \cup B_{-1})$ is $(k-1)$-connected, η' is an isomorphism. Hence $(\varphi_{-1})_\# = 0$.

Hypothesis (b) implies that $\varphi_{k-1} \circ \cdots \circ \varphi_0 : (A_{-1}, B_{-1}) \to (A_{k-1}, B_{k-1})$ is homotopic rel B_{-1} to a map $\psi: (A_{-1}, A_{-1}^{(k-1)} \cup B_{-1}) \to (A_{k-1}, B_{k-1})$. Hence, the following triangle is homotopy commutative:

$$\begin{array}{ccc} (A_{-1}, B_{-1}) & \xrightarrow{\varphi_{k-1} \circ \cdots \circ \varphi_0} & (A_{k-1}, B_{k-1}) \\ \cap & \nearrow \psi & \\ (A_{-1}, A_{-1}^{(k-1)} \cup B_{-1}). & & \end{array}$$

Hence the following diagram commutes:

$$\begin{array}{ccccc} \pi_k'(A_{-2}, B_{-2}) & \xrightarrow{(\varphi_{-1})_\#} & \pi_k'(A_{-1}, B_{-1}) & \xrightarrow{(\varphi_{k-1} \circ \cdots \circ \varphi_0)_\#} & \pi_k'(A_{k-1}, B_{k-1}) \\ & 0 \searrow & \downarrow \subset_\# & \nearrow \psi_\# & \\ & & \pi_k'(A_{-1}, A_{-1}^{(k-1)} \cup B_{-1}). & & \end{array}$$

Thus, $(\varphi_{k-1} \circ \cdots \circ \varphi_0 \circ \varphi_{-1})_\# : \pi_k'(A_{-2}, B_{-2}) \to \pi_k'(A_{k-1}, B_{k-1})$ is the zero homomorphism.

Hypothesis (c) implies that $(\varphi_k)_\#$ sends each element of the form (*) in $\pi_k(A_{k-1}, B_{k-1})$ to zero in $\pi_k(A_k, B_k)$. So $(\varphi_k)_\#$ maps the kernel of the quotient map $q_{k-1} : \pi_k(A_{k-1}, B_{k-1}) \to \pi_k'(A_{k-1}, B_{k-1})$ to zero in $\pi_k(A_k, B_k)$. Consequently, there is a homomorphism $\chi : \pi_k'(A_{k-1}, B_{k-1}) \to \pi_k(A_k, B_k)$ which makes the following triangle commute:

$$\begin{array}{ccc} \pi_k(A_{k-1}, B_{k-1}) & \xrightarrow{(\varphi_k)_\#} & \pi_k(A_k, B_k) \\ q_{k-1} \downarrow & \nearrow \chi & \\ \pi_k'(A_{k-1}, B_{k-1}). & & \end{array}$$

Hence, we have the following commutative diagram:

$$\begin{array}{ccccc} \pi_k(A_{-2},B_{-2}) & \xrightarrow{(\varphi_{k-1}\circ\cdots\circ\varphi_{-1})_\#} & \pi_k(A_{k-1},B_{k-1}) & \xrightarrow{(\varphi_k)_\#} & \pi_k(A_k,B_k) \\ q_{-2}\downarrow & & q_{k-1}\downarrow & \nearrow \chi & \\ \pi'_k(A_{-2},B_{-2}) & \xrightarrow[0]{} & \pi'_k(A_{k-1},B_{k-1}). & & \end{array}$$

We conclude that $(\varphi_k\circ\cdots\circ\varphi_0\circ\varphi_{-1})_\#\colon \pi_k(A_{-2},B_{-2})\to\pi_k(A_k,B_k)$ is zero. □

LEMMA 4.4. *For each* $k\geq 0$, *there exists an* $l\geq 0$ *such that the inclusion-induced homomorphism*

$$\pi_k(V_i-N_{j+1},V_i-V_j)\to\pi_k(V_i-N_{j+l+1},V_i-V_{j+l})$$

is trivial whenever $1\leq i<j$.

Proof. $\pi_0(V_i-N_{j+1},V_i-V_j)=0$ because neither N_{j+1} nor V_j separates V_i. Hence, in the case $k=0$, $l=0$ works.

Observe that $\pi_1(V_i-N_{j+2},V_i-V_{j+1})=0$. Indeed, V_{j+1} collapses onto U'_{2j+1}, $U'_{2j+1}\subset U_{2j+1}$, U_{2j+1} deforms onto a 1-sphere in U_{2j}, and $U_{2j}\subset U_{2j-1}\subset V_{j-1}\subset V_i$. So $\pi_1(V_i,V_i-V_{j+1})=0$ by [10, Lemma 1]. Since the core of N_{j+2} is codimension 3, $\pi_1(V_i-N_{j+2},V_i-V_{j+1})=0$. It follows trivially that

$$\pi_1(V_i-N_{j+1},V_i-V_j)\to\pi_1(V_i-N_{j+2},V_i-V_{j+1})$$

is the zero homomorphism. So in the case $k=1$, $l=1$ works.

We prove the case $k\geq 2$ by induction. Let $0\leq i<j$. Let $p\colon \tilde{V}_i\to V_i$ denote the universal cover, and let $\tilde{Z}$ denote $p^{-1}(Z)$ whenever Z is a subset of V_i. Set $(A_{-2},B_{-2})=(\tilde{V}_i-\tilde{N}_{j+1},\tilde{V}_i-\tilde{V}_j)$. We will argue the existence of nonnegative integers $l(-1)\leq l(0)\leq\cdots\leq l(k)$ so that if we set $(A_r,B_r)=(\tilde{V}_i-\tilde{N}_{j+l(r)+1},\tilde{V}_i-\tilde{V}_{j+l(r)})$ and let $\varphi_r\colon (A_{r-1},B_{r-1})\to(A_r,B_r)$ denote inclusion for $-1\leq r\leq k$, then the hypotheses of Lemma 4.3 will be satisfied.

To begin, let $l(-1)=2$ and consider the following commutative rectangle:

$$\begin{array}{ccc} H_k(A_{-2},B_{-2}) & \xrightarrow{(\varphi_{-1})_*} & H_k(A_{-1},B_{-1}) \\ \downarrow & & \downarrow \\ \bar{H}_c^{4-k}(\tilde{V}_j,\tilde{N}_{j+1}) & \longrightarrow & \bar{H}_c^{4-k}(\tilde{V}_{j+2},\tilde{N}_{j+3}). \end{array}$$

The vertical arrows are the isomorphisms provided by Alexander duality [7, Thm. 6.9.10]. The bottom horizontal arrow is zero because the homotopy of Lemma 4.1 lifts to a proper homotopy in $\tilde{V}_j$. Hence $(\varphi_{-1})_*=0$.

Next, if $0\leq r\leq k-1$ and $l(r-1)$ is already determined, then the inductive hypothesis implies there is an integer $l(r)\geq l(r-1)$ such that $(\varphi_r)_\#\colon \pi_r(A_{r-1},B_{r-1})\to\pi_r(A_r,B_r)$ is zero.

Finally, suppose $l(k-1)$ is already determined. Let $l(k)=l(k-1)+2$, and consider the following commutative rectangle:

$$\begin{array}{ccc} \pi_2(\tilde{V}_i,B_{k-1}) & \longrightarrow & \pi_2(\tilde{V}_i,B_k) \\ \downarrow & & \downarrow \\ \pi_1(B_{k-1}) & \xrightarrow[(\varphi_k)_\#]{} & \pi_1(B_k). \end{array}$$

The vertical arrows are boundary homomorphisms which are onto because $\tilde{V}_i$ is simply connected. (This is the only point at which we need $\tilde{V}_i$ instead of V_i.) Lemma 4.2 implies that the top horizontal arrow is zero. Hence $(\varphi_k)_\# = 0$.

Now Lemma 4.3 implies $(\varphi_k \circ \cdots \circ \varphi_0 \circ \varphi_{-1})_\#: \pi_k(A_{-2}, B_{-2}) \to \pi_k(a_k, B_k)$ is zero. If we set $l = l(k)$, then we obtain the conclusion of Lemma 4.4. □

LEMMA 4.5. *If $1 \le i < j$, then the inclusion-induced homomorphism $\pi_1(N_j) \to \pi_1(V_i)$ is injective, and the inclusion-induced homomorphism $\pi_1(V_{j+1}) \to \pi_1(V_i)$ has image isomorphic to* $\mathbf{Z}$.

Proof. In this proof all unlabelled arrows are inclusion-induced homomorphisms. (5) implies $g_i | K_{2i+1} \times \{1\}: K_{2i+1} \times \{1\} \to K_{2i-1}$ is not homotopically trivial. So, by (15), $\pi_1(g_i'(K_{2i+1} \times \{1\})) \to \pi_1(h_i^{-1}(N_i))$ is injective. Hence $\pi_1(N_{i+1}) \to \pi_1(N_i)$ is injective, so $\pi_1(N_j) \to \pi_1(N_i)$ is injective for $j > i$. According to Lemma 4.1, $\rho: V_i \to N_i$ restricts to the identity on N_{i+1}. So for $i<j$, $\pi_1(N_j) \to \pi_1(N_i)$ equals the composition of $\pi_1(N_j) \to \pi_1(V_i)$ and $\rho_\#$: $\pi_1(V_i) \to \pi_1(N_i)$. Thus, $\pi_1(N_j) \to \pi_1(V_i)$ must be injective for $j > i$.

Lemma 4.1 implies that the inclusion $V_{j+1} \to V_i$ is homotopic to the composition of $\rho: V_{j+1} \to N_{j+1}$ and the inclusion $N_{j+1} \to V_i$. Since $\pi_1(N_{j+1}) \approx \mathbf{Z}$, the image of $\pi_1(V_{j+1}) \to \pi_1(V_i)$ is either 0 or isomorphic to $\mathbf{Z}$. According to the preceding paragraph, $\pi_1(N_{j+2}) \to \pi_1(V_{j+1}) \to \pi_1(V_i)$ is injective. Hence $\pi_1(V_{j+1}) \to \pi_1(V_i)$ is not zero. □

LEMMA 4.6. *For each $k \ge 0$ there exists an $l \ge 0$ such that the inclusion-induced homomorphism*

$$\pi_k(V_i - N_{j+1}, N_{i+1} - N_{j+1}) \to \pi_k(V_{i-l} - N_{j+1}, N_{i-l+1} - N_{j+1})$$

is trivial whenever $l+1 \le i < j$.

Proof. $\pi_0(V_i - N_{j+1}, N_{i+1} - N_{j+1}) = 0$ because N_{j+1} doesn't separate either V_i or N_{i+1}. Hence, in the case $k=0$, $l=0$ works.

Consider the following commutative rectangle, in which the arrows are all inclusion induced homomorphisms:

$$\begin{array}{ccc} \pi_1(V_i - N_{j+1}, N_{i+1} - N_{j+1}) & \longrightarrow & \pi_1(V_{i-2} - N_{j+1}, N_{i-1} - N_{j+1}) \\ \downarrow & & \downarrow \\ \pi_1(V_i, N_{i+1}) & \longrightarrow & \pi_1(V_{i-2}, N_{i-2}). \end{array}$$

Lemma 4.1 implies that the lower horizontal arrow is the zero homomorphism. The vertical arrows are isomorphisms because N_{j+1} has a codimension-3 spine. So the upper horizontal arrow is the zero homomorphism. Hence, in the case $k=1$, $l=2$ works.

We prove the case $k \ge 2$ by induction. Let $i<j$, where i is large relative to the l's determined in previous steps of the proof. Suppose $0 = l(-2) \le l(-1) \le \cdots \le l(k)$ are integers. For $-2 \le r \le k$, let $p_r: \tilde{V}_{i-l(r)} \to V_{i-l(r)}$ be the universal cover, and set

$$A_r = p_r^{-1}(V_{i-l(r)} - N_{j+1}) \quad \text{and} \quad B_r = p_r^{-1}(N_{i-l(r)+1} - N_{j+1}).$$

Then for $-1 \leq r \leq k$, the inclusion $V_{i-l(r-1)} \to V_{i-l(r)}$ lifts to a map $\tilde{V}_{i-l(r-1)} \to \tilde{V}_{i-l(r)}$ which restricts to a map $\varphi_r: (A_{r-1}, B_{r-1}) \to (A_r, B_r)$. We will argue that the integers $l(-1), l(0), \ldots, l(k)$ can be chosen so that the hypotheses of Lemma 4.3 are satisfied.

To begin, let $l(-1)=2$ and consider the following commutative rectangle:

$$\begin{array}{ccc} H_k(A_{-2}, B_{-2}) & \xrightarrow{(\varphi_{-1})_*} & H_k(A_{-1}, B_{-1}) \\ \downarrow & & \downarrow \\ H_k(\tilde{V}_j, p_{-2}^{-1}(N_{j+1})) & \longrightarrow & H_k(\tilde{V}_{j-2}, p_{-1}^{-1}(N_{j-1})). \end{array}$$

In this diagram, the vertical arrows are excision isomorphisms. The bottom horizontal arrow is induced by the lift of the inclusion $V_i \to V_{i-2}$. Since the homotopy of Lemma 4.1 lifts to a homotopy in $\tilde{V}_{i-2}$, the bottom horizontal arrow must be zero. Hence $(\varphi_{-1})_* = 0$.

Next, if $0 \leq r \leq k-1$ and $l(r-1)$ is already determined, then the inductive hypothesis implies there is an integer $l(r) \geq l(r-1)$ such that $(\varphi_r)_\#$: $\pi_r(A_{r-1}, B_{r-1}) \to \pi_r(A_r, B_r)$ is zero.

Finally, suppose $l(k-1)$ is already determined. Let $s = i - l(r-1)$. Lemma 4.5 implies that $\pi_1(N_{s+1}) \to \pi_1(V_s)$ is injective. Since N_{j+1} has a codimension-3 spine, $\pi_1(N_{s+1} - N_{j+1}) \to \pi_1(V_s)$ is injective. Hence, the lift $\pi_1(B_{k-1}) \to \pi_1(\tilde{V}_s)$ is injective. Since $\tilde{V}_s$ is simply connected, we have $\pi_1(B_{k-1}) = 0$. (This is the only point at which we need $\tilde{V}_i$ instead of V_i.) So, if we set $l(k) = l(k-1)$, then $(\varphi_k)_\#$: $\pi_1(B_{k-1}) \to \pi_1(B_k)$ is zero.

Now Lemma 4.3 implies $(\varphi_k \circ \cdots \circ \varphi_0 \circ \varphi_{-1})_\#$: $\pi_k(A_{-2}, B_{-2}) \to \pi_k(A_k, B_k)$ is zero. If we set $l = l(k)$ then we obtain the conclusion of Lemma 4.6. □

5. Controlled Embedding and *h*-Cobordism Theorems

Our proof of the converse direction of Theorem 1 is based on a variant of Theorem 7.2C (technical controlled *h*-cobordism) of [3]. Unfortunately, this result is not technical enough for our purposes. Fortunately, we can vary this result to meet our needs, and the proof of the variant can be obtained from the proof in [3] merely by modifying the logical outline. No new topological ideas are needed.

Theorem 7.2C of [3] depends on the controlled embedding theorem in Section 5.4 of [3]. Both these results involve a control map $p: E \to Z$ on which the hypothesis of "good" local fundamental groups has been imposed. This means that every neighborhood U of a point $z \in Z$ must contain a neighborhood V of z such that the image of $\pi_1(p^{-1}(V)) \to \pi_1(p^{-1}(U))$ belongs to the class of "good" groups. Here, "good" stands for either "poly-(finite or cyclic)" or "torsion free poly-(finite or cyclic)". This hypothesis is too strong for our situation. We weaken it in two different ways. First, and more significantly, we don't allow the U's to range over arbitrarily small neighborhoods; instead, we impose a lower bound on their diameters. Second, we don't allow the z's to range over all of Z; they are restricted to lie in a subset Y of Z because we don't intend to assert control over $Z - Y$.

The following definition makes our hypothesis on $p: E \to Z$ precise. Let $p: E \to Z$ be a map between compact metric spaces, let $\mathcal{G}$ be a class of groups, let $\alpha > 0$ and $\beta > 0$, and let $Y \subset Z$. We say that $p: E \to Z$ has *α, β fundamental groups in $\mathcal{G}$ over Y* if every subset A of Z of diameter $< \alpha$ which intersects Y is contained in a subset B of Z of diameter $< \beta$ such that the image of $\pi_1(p^{-1}(A)) \to \pi_1(p^{-1}(B))$ belongs to $\mathcal{G}$.

To state controlled embedding and h-cobordism theorems that suit our needs, we introduce some new terminology. Some of these definitions are simply relativizations of definitions appearing in [3]. Others are new terms introduced to simplify the statements of our theorems.

Let Z be a metric space, let $Y \subset Z$, and let $p: E \to Z$ be a map. (Z is the *control space* and p is the *control map*.) Let $f: W \to E$ be a map. The *Z-diameter* of a subset S of W means the diameter of $p \circ f(S)$ in Z. A subset of W is *over Y* if it is contained in $f^{-1}(p^{-1}(Y))$. Let $\delta > 0$. $f: W \to E$ is *δ-1-connected over Y* if, given a relative 2-complex (K, L) and maps $\varphi: L \to W$ and $\psi: K \to p^{-1}(Y)$ such that $f \circ \varphi = \psi | L$, there is a map $\Phi: K \to W$ which extends φ such that $p \circ f \circ \Phi$ is within δ of $p \circ \psi$ in Z. In this situation, we call Φ an *approximate lift* of ψ with *Z-error* $< \delta$. A homotopy $h: A \times [0,1] \to W$ is a *δ-homotopy over Z* if $h(\{a\} \times [0,1])$ has Z-diameter $< \delta$ for each $a \in A$. Two maps into W are *δ-homotopic over Z* if they are joined by a δ-homotopy over Z. Suppose (W, M_0, M_1) is a cobordism. $f: W \to E$ is a *δ-h-cobordism over Y* if, for $i = 0, 1$, $\mathrm{id}_{f^{-1}(p^{-1}(Y))}$ is δ-homotopic over Z to a map from $f^{-1}(p^{-1}(Y))$ into M_i. $f: W \to E$ has a *δ-product structure over Y* if there is an embedding $e: M_0 \times [0,1] \to W$ such that, for every $x \in M_0$, $e(x, 0) = x$, $e(\{x\} \times [0,1])$ has Z-diameter $< \delta$, $e(\partial M_0 \times [0,1]) \subset \partial W$, and $f^{-1}(p^{-1}(Y)) \subset e(M_0 \times [0,1])$. Let $Y^{(\delta)}$ to denote the δ-neighborhood of Y in Z: $Y^{(\delta)} = \{z \in Z:$ the distance from z to some point of Y is $< \delta\}$.

Again let Z be a metric space, and let $p: E \to Z$ be a map. Let M be a 4-manifold, and let $f: M \to E$ be a map. Suppose A is the union of finitely many disjoint disks, and $\alpha: A \to M$ is a map. Let $\delta > 0$. α is a *well-equipped δ-embedding* if α is an embedding such that, for each component D of A, $\alpha(D)$ has Z-diameter $< \delta$ and $\alpha(A)$ has an immersed transverse sphere of Z-diameter $< \delta$ which intersects $\alpha(D)$. α is a *well-equipped δ-immersion* if α is an immersion in which the image of each component of A has Z-diameter $< \delta$, α has δ-algebraically transverse spheres, and the images of distinct components of A have δ-algebraically trivial intersections. (Definitions of "δ-algebraically transverse spheres" and "δ-algebraically trivial intersections" appear on page 90 of [3]. Be aware that in these definitions "diameter" always means "Z-diameter".)

We now state variants of the controlled embedding and h-cobordism theorems of [3] which are tailored to our needs.

THEOREM 5.1 (Controlled embedding). *Let $\mathcal{G}$ be the class of all poly-(finite or cyclic) groups, and let Z be a compact metric space. Then for every $\epsilon > 0$ there is a $\beta = \beta(Z, \epsilon)$, and for every $\alpha > 0$ there is a $\delta = \delta(Z, \alpha)$, with the following property. Let $Y \subset Z$, and suppose $\alpha < \beta$ and $p: E \to Z$ is a map between* *compact metric spaces which has α, β fundamental groups in $\mathcal{G}$ over $Y^{(\epsilon)}$. Let M be a 4-manifold, and let $f: M \to E$ be a map which is δ-1-connected over $Y^{(\epsilon)}$. If A is the union of finitely many disjoint disks, then for every well-equipped δ-immersion of A in $f^{-1}(p^{-1}(Y))$ there is a well-equipped (topological) ϵ-embedding of A in $f^{-1}(p^{-1}(Y^{(\epsilon)}))$ with the same framed boundary.*

THEOREM 5.2 (Controlled h-cobordism). *Let $\mathcal{G}$ be the class of all torsion free poly-(finite or cyclic) groups, and let Z be a compact polyhedron equipped with a metric. Then for every $\epsilon > 0$ there is a $\beta = \beta(Z, \epsilon)$, and for every $\alpha > 0$ there is a $\delta = \delta(Z, \alpha)$, with the following property. Let $Y \subset Z$, and suppose $\alpha < \beta$ and $p: E \to Z$ is a piecewise linear map between compact polyhedra which has α, β fundamental groups in $\mathcal{G}$ over $Y^{(\epsilon)}$. If (W, M_0, M_1) is a 5-dimensional cobordism and $f: W \to E$ is a δ-1-connected δ-h-cobordism over $Y^{(\epsilon)}$, then $f: W \to E$ has an ϵ-product structure over Y.*

We remark that for simplicity we have backed away from the generality of Theorem 7.2C of [3] in several respects. We have assumed Z to be compact, which allows us to regard ϵ, β, α, and δ as constants rather than functions. We have taken $p: E \to Z$ to be a piecewise linear map between polyhedra instead of a simplicial NDR [3, p. 108]. Our assumption that the groups in $\mathcal{G}$ are torsion free kills the obstruction groups which would otherwise arise. (See the remark preceding Corollary 7.2B on page 109 of [3].) We have not required the ϵ-product structure over Y to agree with a previously existing δ-product structure over a subset of Y. Also, we have not insisted that the ϵ-product structure be smooth off a regular neighborhood of a 1- or 2-complex. The greater generality could be introduced in Theorems 5.1 and 5.2 at the cost of lengthening and complicating the proof in a routine way.

We now explain how to extract proofs of Theorems 5.1 and 5.2 from the proofs of their precursors in [3]. The proofs of the controlled embedding and h-cobordism theorems in [3] are logically similar in that each begins with the given ϵ and works "backwards" through a finite number (say n) of steps to find δ. These n steps generate a finite sequence of positive numbers $\epsilon = \gamma_0, \gamma_1, \ldots, \gamma_n$. The proof is completed by setting $\delta = \gamma_n$. For $1 \leq i \leq n$, the ith step consists of a proof of a statement of the form: "For every $\gamma_{i-1} > 0$, there is a $\gamma_i > 0$ such that if certain subsets in the given situation all have Z-diameter $< \gamma_i$, then the situation can be improved in some desirable manner after which certain subsets all have Z-diameter $< \gamma_{i-1}$." A point of fundamental importance in both proofs is that the determination of γ_i from γ_{i-1} is based solely on the topology and metric on Z, and is independent of the other spaces or maps (such as Y, $p: E \to Z$, $f: M \to E$, or $f: W \to E$) appearing in the theorem.

In the proof of the controlled embedding theorem in [3], there is only one step which invokes the hypothesis that $p: E \to Z$ has local fundamental groups in $\mathcal{G}$. Suppose this is the jth step. Then our proof of Theorem 5.1 uses steps 1 through $j-1$ of the proof in [3] to determine $\beta = \beta(Z, \epsilon)$ from ϵ; and it uses steps $j+1$ through n to determine $\delta = \delta(Z, \alpha)$ from α. In step j, our proof replaces the appeal to the local fundamental groups hypothesis by an appeal to the hypothesis that $p: E \to Z$ has α, β fundamental groups in $\mathcal{G}$ over Y. More explicitly, beginning with $\gamma_0 = \epsilon$, steps 1 through $j-1$ of the proof in [3] generate the sequence $\epsilon = \gamma_0 \mapsto \gamma_1 \mapsto \cdots \mapsto \gamma_{j-1}$ without invoking the local fundamental groups hypothesis. We repeat these steps in our proof of Theorem 5.1 and conclude by setting $\beta(Z, \epsilon) = \gamma_{j-1}/3$. Similarly, beginning with $\gamma_j = \alpha$, steps $j+1$ through n of the proof in [3] generate a sequence $\alpha = \gamma_j \mapsto \gamma_{j+1} \mapsto \cdots \mapsto \gamma_n$ without invoking the local fundamental groups hypothesis. Again, we repeat these steps in our proof and conclude by setting $\delta(Z, \alpha) = \gamma_n$.

To complete our explanation of the proof of Theorem 5.1, we must tell how to modify the crucial "jth step" of the proof of the controlled embedding theorem in [3] in which the local fundamental groups hypothesis appears. This step establishes the statement: "For every $\gamma_{j-1} > 0$, there is a $\gamma_j > 0$ such that if there is a size-γ_j disklike capped grope of height ≥ 4 in M over $Y^{(\gamma_j)}$ with disjoint component images, then there is a (γ_{j-1})-π_1-*null* size-γ_{j-1} disklike capped grope of height ≥ 4 in M over $Y^{(\gamma_{j-1})}$ with disjoint component images and with the same framed boundary." (We say that a capped grope with transverse spheres is *size-*γ if the image of each component of the grope has Z-diameter $< \gamma$ and each transverse sphere has Z-diameter $< \gamma$. In addition, the grope is γ-π_1-*null* if each loop in the image of the grope is null homotopic in a set of Z-diameter $< \gamma$.) The proof in [3] of this statement first uses the hypothesis (unavailable to us) that $p: E \to Z$ has local fundamental groups in $\mathcal{G}$ to find $\gamma_j > 0$, so that every subset A of Z of diameter $< \gamma_j$ is contained in a subset B of Z of diameter $< \gamma_{j-1}/3$ such that the image of $\pi_1(p^{-1}(A)) \to \pi_1(p^{-1}(B))$ belongs to $\mathcal{G}$. Then the proof invokes Proposition 2.9 of [3] in a small neighborhood of each component of the given size-γ_j capped grope over $Y^{(\gamma_j)}$ to get a size-$(\gamma_{j-1}/3)$ capped grope over $Y^{(\gamma_{j-1}/3)}$ with the property that each loop in its image becomes null homotopic in a set of Z-diameter $< \gamma_{j-1}/3$ when mapped into E. Finally, the hypothesis that $f: M \to E$ is δ-1-connected where $\delta < \gamma_j < \gamma_{j-1}/3$ is invoked to conclude that the size-$(\gamma_{j-1}/3)$ capped grope just obtained is γ_{j-1}-π_1-null. *Our* proof of the jth step first invokes the hypothesis that $\alpha < \beta$ and $p: E \to Z$ has α, β fundamental groups in $\mathcal{G}$ over Y, where $\beta = \gamma_{j-1}/3$. Then we set $\gamma_j = \alpha$, and apply Proposition 2.9 of [3] and the δ-1-connectedness of f as before. This completes the proof of Theorem 5.1.

The proofs of Theorem 7.2C in [3] and of our own Theorem 5.2 have the same basic outline. Throughout the proof one works with a collar $c: M_0 \times [0,1] \to W$ on M_0 and with a handlebody decomposition of $\mathrm{cl}(W - c(M_0 \times [0,1]))$. For $S \subset Z$, the "handles over S" refers to the handles of the decomposition which intersect $f^{-1}(p^{-1}(S))$. We say that the handle decomposition is *size-*γ if γ is an upper bound on the Z-diameters of all handles and collar fibers. Each step of the proof simplifies the handlebody decomposition over Y while stretching the collar over more of $f^{-1}(p^{-1}(Y))$. The ith step of the proof simplifies the handle decomposition over $Y^{(\epsilon - \gamma_{i-1})}$, and transforms the handle decomposition from size-γ_i to size-γ_{i-1}. The last step of the proof, which is the beginning of the process of simplifying the handlebody decomposition, starts out with every collar fiber and every handle of Z-diameter $<\gamma_n=\delta$. In the first step of the proof, which is the end of the process of simplifying the handlebody decomposition, the last remaining handles over Y are cancelled and the collar is stretched over all of $f^{-1}(p^{-1}(Y))$, with all collar fibers of Z-diameter $<\gamma_0=\epsilon$, thus producing an ϵ-product structure over Y.

In the proof of Theorem 7.2C in [3], there is only one step that invokes a local fundamental group hypothesis on the control map. Again, suppose this is the jth step. Then, beginning with $\gamma_0=\epsilon$, steps 1 through $j-1$ of the proof of Theorem 7.2C generate a sequence $\epsilon=\gamma_0\mapsto\gamma_1\mapsto\cdots\mapsto\gamma_{j-1}$ without invoking the local fundamental groups hypothesis. Similarly, steps $j+1$ through n of the proof of Theorem 7.2C generate a sequence $\gamma_j\mapsto\gamma_{j+1}\mapsto\cdots\mapsto\gamma_n=\delta$ without invoking the local fundamental groups hypothesis. Again we repeat these steps in our proof of Theorem 5.2 as part of the process of determining β from ϵ and δ from α. However, here the situation differs slightly from that in Theorem 5.1. Steps 1 through $j-1$ and $j+1$ through n alone are not enough to determine β from ϵ and δ from α. We must delve into the jth step as well.

The jth step of the proof of the controlled h-cobordism theorem in [3] established the statement: "For every $\gamma_{j-1}>0$, there is a $\gamma_j>0$ with the following property. Suppose $f\colon W\to E$ is γ_j-1-connected over $Y^{(\epsilon)}$. Suppose the handle decomposition of W is size-γ_j, all the handles over $Y^{(\epsilon-\gamma_j)}$ are of index 2 or 3, and M is the 'level surface' between the 2- and 3-handles. Suppose that if a 2-handle and a 3-handle intersect and their union intersects $f^{-1}(p^{-1}(Y^{(\epsilon-\gamma_{j-1})}))$, then the excess intersections between the belt sphere of the 2-handle and the attaching sphere of the 3-handle are paired by immersed Whitney disks in M of Z-diameter $<\gamma_j$. Then conclude that all such excess intersections are also paired by *disjoint embedded* Whitney disks in M of Z-diameter $<\gamma_{j-1}$."

If a detailed proof of this statement were given in [3], it would begin by invoking the controlled embedding theorem of Section 5.4 of [3] to obtain a $\delta'>0$ with the property that if $f|M\colon M\to E$ is δ'-1-connected over $Y^{(\epsilon-(\gamma_{j-1}/3))}$, then immersed Whitney disks in $M\cap f^{-1}(p^{-1}(Y^{(\epsilon-(2\gamma_{j-1}/3))}))$ of Z-diameter $<\delta'$ can be replaced by disjoint topologically embedded Whitney disks of Z-diameter $<(\gamma_{j-1})/3$ with the same framed boundary. We choose $\gamma_j=\min\{\delta'/2,\gamma_{j-1}/6\}$. Now suppose $f\colon W\to E$ is γ_j-1-connected over $Y^{(\epsilon)}$, and suppose that the handle decomposition of W is size-γ_j and that all the handles over $Y^{(\epsilon-\gamma_j)}$ are of index 2 or 3. Since $f\colon W\to E$ is γ_j-1-connected over $Y^{(\epsilon-(\gamma_{j-1}/3))}$, and all handles are of Z-diameter $<\gamma_j$, and only handles of index 2 or 3 lie over $Y^{(\epsilon-(\gamma_{j-1}/3)+\gamma_j)}$, it follows that $f|M\colon M\to E$ is $(2\gamma_j)$-1-connected over $Y^{(\epsilon-(\gamma_{j-1}/3))}$. (*Proof:* A map of a 2-complex into $p^{-1}(Y^{(\epsilon-(\gamma_{j-1}/3))})$ can be lifted into W with Z-error $<\gamma_j$. This lift lies in $f^{-1}(p^{-1}(Y^{(\epsilon-(\gamma_{j-1}/3)+\gamma_j)}))$. So it can be pushed off 2- and 3-handles and into M, introducing a further Z-error $<\gamma_j$. Hence, the total Z-error of the lift to

M is $< 2\gamma_j$.) Since $2\gamma_j < \delta'$, the choice of δ' guarantees that immersed Whitney disks of Z-diameter $< \gamma_j$ can be replaced by disjoint topologically embedded Whitney disks of Z-diameter $< \gamma_{j-1}$ with the same framed boundary.

Unfortunately, the controlled embedding theorem in [3] requires the hypothesis (unavailable to us) that $p: E \to Z$ has local fundamental groups in $\mathcal{G}$. At this point in *our* proof, we apply our Theorem 5.1 with the ϵ in the statement of Theorem 5.1 replaced by $\gamma_{j-1}/3$ (with the given α), and with Y replaced by $Y^{(\epsilon-(2\gamma_{j-1}/3))}$. Then Theorem 5.1 provides a $\beta > 0$ depending on Z and $\gamma_{j-1}/3$, and a $\delta' > 0$ depending on Z and α. We may assume $\beta \leq \gamma_{j-1}/3$ and $\delta' \leq \alpha$. We now invoke our hypothesis that $\alpha < \beta$ and $p: E \to Z$ has α, β fundamental groups in $\mathcal{G}$ over $Y^{(\epsilon)}$ to reach the same point that had been attained in the previous paragraph under the good local fundamental groups hypothesis. Specifically, we now know that if $f|M: M \to E$ is δ'-1-connected over $Y^{(\epsilon-(\delta_{j-1}/3))}$, then immersed Whitney disks in $M \cap f^{-1}(p^{-1}(Y^{(\epsilon-(2\gamma_{j-1}/3))}))$ of Z-diameter $< \delta'$ can be replaced by disjoint topologically embedded Whitney disks of Z-diameter $< (\gamma_{j-1})/3$ with the same framed boundary. We then set $\gamma_j = \delta'/2$. Since $\delta'/2 \leq \alpha/2 < \beta/2 \leq \gamma_{j-1}/6$, $\gamma_j \leq \min\{\delta'/2, \gamma_{j-1}/6\}$. So we can now proceed as in the previous paragraph to change immersed Whitney disks to embedded ones.

We observe that β is generated from ϵ via the sequence $\epsilon = \gamma_0 \mapsto \gamma_1 \mapsto \cdots \mapsto \gamma_{j-1} \mapsto \beta$, where the last arrow is provided by Theorem 5.1; hence we can write $\beta = \beta(Z, \epsilon)$. Similarly, $\delta = \gamma_n$ is now generated from α via the sequence $\alpha \mapsto \delta' \mapsto \delta'/2 = \gamma_j \mapsto \gamma_{j+1} \mapsto \cdots \mapsto \gamma_n = \delta$, where the first arrow is provided by Theorem 5.1; thus we can write $\delta = \delta(Z, \alpha)$. This completes our explanation of Theorem 5.2.

We end this section with a lemma which gives a useful condition for detecting that a map is a δ-h-cobordism.

LEMMA 5.1. *For every $\delta > 0$, there is a $\gamma = \gamma(\delta)$ with the following property. Suppose (W, M_0, M_1) is a 5-dimensional cobordism, $f: W \to E$ and $p: E \to Z$ are maps between metric spaces, and $Y \subset Z$. For $A \subset Z$, let $A^* = f^{-1}(p^{-1}(A))$. Then $f: W \to E$ is a δ-h-cobordism over Y if for each $i = 0, 1$ and each $k = 0, 1, \ldots, 5$, the following condition is satisfied:*

(∗) *If $\zeta \leq \delta$, then every subset A of $Y^{(\zeta)}$ is contained in a subset B of $Y^{(\zeta+\gamma)}$ such that* $\operatorname{diam} B < \operatorname{diam} A + \gamma$ *and the inclusion-induced homomorphism $\pi_k(A^*, A^* \cap M_i) \to \pi_k(B^*, B^* \cap M_i)$ is zero.*

Proof. Set $\gamma = \delta/95$. We will construct a δ-homotopy over Z which joins id_{Y^*} to a map from Y^* into M_0. By a similar argument, we can produce a δ-homotopy over Z which joins id_{Y^*} to a map from Y^* into M_1.

Since W is a 5-dimensional ANR, there is a 5-dimensional polyhedron P and maps $\varphi: W \to P$ and $\psi: P \to W$ such that $\psi \circ \varphi$ is γ-homotopic over Z to id_W. There is a subpolyhedron Q of P such that $\varphi(Y^*) \subset Q \subset \psi^{-1}((Y^{(\gamma)})^*)$. Triangulate P so that Q is a subcomplex and so that for each simplex σ of P, $\psi(\sigma)$ has Z-diameter $< \gamma$. For $0 \leq k \leq 5$, let Q^k denote the k-skeleton of Q.

Set $a_0 = 1$ and inductively define $a_k = 2a_{k-1} + 2$ for $1 \le k \le 5$. Then $a_5 = 94$. Set $b_k = 2 + k$ for $0 \le k \le 5$. For each k, $0 \le k \le 5$, we will inductively construct an $a_k\gamma$-homotopy over Z, denoted $\Psi_k: Q^k \times [0,1] \to (Y^{(b_k\gamma)})^*$, such that $\Psi_k(x,0) = \psi(x)$ and $\Psi_k(x,1) \in M_0$ for each $x \in Q^k$.

First we construct Ψ_0. Let $v \in Q^0$. Set $A = \{p \circ f \circ \psi(v)\}$. Then $A \subset Y^{(\gamma)}$. So condition (*) (with $i = 0$ and $k = 0$) implies that A is contained in a subset B of $Y^{(2\gamma)} = Y^{(b_0\gamma)}$ such that $\operatorname{diam} B < \operatorname{diam} A + \gamma = \gamma$, and there is a path $\Psi_{0,v}: \{v\} \times [0,1] \to B^*$ such that $\Psi_{0,v}(v,0) = v$ and $\Psi_{0,v}(v,1) \in M_0$. Now define $\Psi_0: Q^0 \times [0,1] \to Y^{(b_0\gamma)}$ by $\Psi_0 = \bigcup_{v \in Q^0} \Psi_{0,v}$. Then Ψ_0 is the desired $a_0\gamma$-homotopy over Z.

Next let $1 \le k \le 5$ and inductively assume that $\Psi_{k-1}: Q^{k-1} \times [0,1] \to (Y^{(b_k\gamma)})^*$ is an $a_{k-1}\gamma$-homotopy over Z such that for each $x \in Q^{k-1}$, $\Psi_{k-1}(x,0) = \psi(x)$ and $\Psi_{k-1}(x,1) \in M_0$. Let σ be a k-simplex of Q. Set $A = p \circ f \circ (\psi(\sigma) \cup \Psi_{k-1}(\partial\sigma \times [0,1]))$. Then $A \subset Y^{(b_{k-1}\gamma)}$ and $\operatorname{diam} A < (2a_{k-1} + 1)\gamma$. If we regard $(\psi \,|\, \sigma) \cup (\Psi_{k-1} \,|\, \partial\sigma \times [0,1])$ as a representative of an element of $\pi_k(A^*, A^* \cap M_0)$, then condition (*) implies that A is contained in a subset B of $Y^{((b_{k-1}+1)\gamma)} = Y^{(b_k\gamma)}$ such that $\operatorname{diam} B < \operatorname{diam} A + \gamma = (2a_{k-1} + 2)\gamma = a_k\gamma$, and $\Psi_{k-1} \,|\, \partial\sigma \times [0,1]$ extends to a map $\Psi_{k,\sigma}: \sigma \times [0,1] \to B^*$ such that for each $x \in \sigma$, $\Psi_{k,\sigma}(x,0) = \psi(x)$ and $\Psi_{k,\sigma}(x,1) \in M_0$. Now define $\Psi_k: Q^k \times [0,1] \to (Y^{(b_k\gamma)})^*$ by $\Psi_k = \bigcup\{\Psi_{k,\sigma}: \sigma$ is a k-simplex of $Q\}$. Then Ψ_k is the desired $a_k\gamma$-homotopy over Z.

Finally, $\Psi_5 \circ ((\varphi \,|\, Y^*) \times \mathrm{id}_{[0,1]})$ is a 94γ-homotopy over Z joining $\psi \circ \varphi \,|\, Y^*$ to a map from Y^* into M_0. If we follow the γ-homotopy over Z joining id_{Y^*} to $\psi \circ \varphi \,|\, Y^*$ with $\Psi_5 \circ ((\varphi \,|\, Y^*) \times \mathrm{id}_{[0,1]})$, we obtain a δ-homotopy over Z which joins id_{Y^*} to a map from Y^* into M_0. □

6. Conclusion of the Proof of Theorem 1

In this section we complete the proof of the converse direction of Theorem 1. The proof is based on Theorem 5.2 (Controlled h-cobordism). The idea is to stretch a controlled product structure over a portion of $M \times [0,1]$ so that for some $i \ge 1$, each fiber which originates in $N_i \times \{0\}$ ends in $U \times \{1\}$ and each point of $X \times \{1\}$ is the endpoint of a fiber which originates in $(\operatorname{int} N_i) \times \{0\}$. Then the product structure determines a homeomorphism from N_i to a neighborhood N of X in U. Since M is orientable and N_i is a regular neighborhood of a 1-sphere, $N \cong S^1 \times B^3$.

We apply Theorem 5.2 with the control space $Z = [0,1]$ and with $\epsilon = \frac{1}{4}$. Then Theorem 5.2 provides a $\beta > 0$. We set $\alpha = \min\{\beta/2, \frac{1}{8}\}$. Then Theorem 5.2 provides a $\delta > 0$. We can assume $\delta < \frac{1}{8}$. Finally, with this δ as input, Lemma 5.1 provides a $\gamma > 0$.

Other variables in the statement of Theorem 5.2 are specified as follows: $Y = [\frac{1}{2}, 1]$, $(W, M_0, M_1) = (M \times [0,1], M \times \{0\}, M \times \{1\})$, $E = M \times [0,1]$, and $f = \mathrm{id}_{M \times [0,1]}$. The remaining variable is the control map $p: E \to Z$. Its specification is more complicated.

Let $l \ge 2$ be an upper bound of all the integers l which appear in the statements of Lemmas 4.4 and 4.6 as k ranges from 0 to 5. Let n be a multiple of 4 which is an upper bound of $\{6/\alpha, 8(l+2), (l+4)/\gamma\}$. $V_0, V_1, \ldots, V_n$ and $N_1, N_2, \ldots, N_{n+1}$ are the PL submanifolds of M specified in Section 3. For $0 \le i \le n$, define the PL submanifold W_i of $M \times [0,1]$ by $W_0 = V_0 \times [0,1]$ and $W_i = (N_{i+1} \times [0, 1-(1/i)]) \cup (V_i \times [1-(1/i), 1])$ for $1 \le i \le n$. Then $W_i \subset \operatorname{int} W_{i-1}$ for $1 \le i \le n$. For $0 \le i < j \le n$, define $W(i,j) = \operatorname{cl}(W_i - W_j)$. Finally, choose the control map $p: M \times [0,1] \to [0,1]$ to be any piecewise linear map such that $p(\operatorname{cl}(M \times [0,1] - W_0)) = \{0\}$, $p(W(i-1, i)) = [(i-1)/n, i/n]$ for $1 \le i \le n$, and $p(W_n) = \{1\}$.

We must verify the hypotheses of Theorem 5.2. First, we argue that over $Y^{(\epsilon)} = [\frac{1}{4}, 1]$, the control map $p: M \times [0,1] \to [0,1]$ has α, β fundamental groups in the class $\mathcal{G}$ of all torsion free poly-(finite or cyclic) groups. We begin by observing that if $3 \le i < j \le n$, then the inclusion-induced homomorphisms $\pi_1(W(i,j)) \to \pi_1(W(i-2,j))$ and $\pi_1(W_i) \to \pi_1(W_{i-2})$ have images isomorphic to $\mathbf{Z}$. To see this, let $t = 1-(1/i)$ and consider the following two commutative diagrams of inclusion-induced homomorphisms:

$$\begin{array}{ccc}
\pi_1(W(i,j)) & \longrightarrow & \pi_1(W(i-2,j)) \\
\uparrow & & \uparrow \\
\pi_1(\operatorname{cl}(V_i - N_{j+1}) \times \{t\}) & \longrightarrow & \pi_1(\operatorname{cl}(V_{i-2} - N_{j+1}) \times \{t\}) \\
\downarrow & & \downarrow \\
\pi_1(V_i \times \{t\}) & \longrightarrow & \pi_1(V_{i-2} \times \{t\});
\end{array}$$

$$\begin{array}{ccc}
\pi_1(W_i) & \longrightarrow & \pi_1(W_{i-2}) \\
\uparrow & & \uparrow \\
\pi_1(V_i \times \{t\}) & \longrightarrow & \pi_1(V_{i-2} \times \{t\}).
\end{array}$$

In both diagrams, the vertical upward pointing arrows are isomorphisms because the inclusions that induce them are homotopy equivalences. In the first diagram, the vertical downward pointing arrows are isomorphisms because N_{j+1} has a codimension-3 spine. Lemma 4.5 implies that the bottom horizontal arrows in both diagrams have images isomorphic to $\mathbf{Z}$. Hence, the top horizontal arrows in both diagrams have images isomorphic to $\mathbf{Z}$. Now let A be a subset of $[0,1]$ of diameter $< \alpha$ which intersects $[\frac{1}{4}, 1]$. Since $\alpha \le \frac{1}{8}$, $A \subset [\frac{1}{8}, 1]$; and since $n \ge 8(l+2) \ge 32$, there are integers $4 \le i < j \le n$ such that $A \subset [i/n, j/n]$ and $(j-i)/n < \alpha + (2/n)$. First consider the case $j < n-1$. Since $A \subset ((i-1)/n, (j+1)/n)$, $p^{-1}(A) \subset W(i-1, j+1)$. Set $B = [(i-3)/n, (j+1)/n]$. Then $\operatorname{diam} B = (j-i)/n + 4/n < \alpha + (6/n) \le 2\alpha \le \beta$, and $W(i-3, j+1) \subset p^{-1}(B)$. Since the image of

$$\pi_1(W(i-1, j+1)) \to \pi_1(W(i-3, j+1))$$

is isomorphic to $\mathbf{Z}$, the image of $\pi_1(p^{-1}(A)) \to \pi_1(p^{-1}(B))$ is isomorphic to a subgroup of $\mathbf{Z}$ and thus is an element of $\mathcal{G}$. Now consider the case $n-1 \le j \le n$. Since $A \subset ((i-1)/n, 1]$, $p^{-1}(A) \subset W_{i-1}$. Set $B = [(i-3)/n, 1]$. Then $\operatorname{diam} B = (n-i)/n + 3/n \le (j-i)/n + 4/n < \beta$, and $W_{i-3} \subset p^{-1}(B)$. Since the image of $\pi_1(W_{i-1}) \to \pi_1(W_{i-3})$ is isomorphic to $\mathbf{Z}$, the image of $\pi_1(p^{-1}(A)) \to \pi_1(p^{-1}(B))$ is isomorphic to a subgroup of $\mathbf{Z}$ and thus is an element of $\mathcal{G}$.

Second, we note that $f: W \to E$ is obviously δ-1-connected because $f = id_{M\times[0,1]}$.

Third, we must verify that $f: W \to E$ is a δ-h-cobordism over $Y^{(\epsilon)} = [\frac{1}{4}, 1]$. We will achieve this by invoking Lemma 5.1. Hence, it suffices to prove that for $t=0,1$ and $0 \le k \le 5$, if $\zeta \le \delta$ then every $A \subset [\frac{1}{4},1]^{(\zeta)}$ is contained in a $B \subset [\frac{1}{4},1]^{(\zeta+\gamma)}$ such that $\operatorname{diam} B < \operatorname{diam} A + \gamma$ and $\pi_k(A^*, A^* \cap (M\times\{t\})) \to \pi_k(B^*, B^*\cap(M\times\{t\}))$ is zero. We first introduce the following abbreviations. For $1 \le i < j \le n$ and $t \in [0,1]$, let $(W_i)_t = W_i \cap (M\times\{t\})$ and $W(i,j)_t = W(i,j)\cap(M\times\{t\})$, and let $M(i,j) = \operatorname{cl}(V_i - N_{j+1})$, $V(i,j) = \operatorname{cl}(V_i - V_j)$, and $N(i,j) = \operatorname{cl}(N_{i+1} - N_{j+1})$.

We begin with the following observations.

(1) $\pi_k(W(i,j), W(i,j)_0) \to \pi_k(W(i-l,j), W(i-l,j)_0)$ is zero for $l+1 \le i < j \le n$ and $0 \le k \le 5$.
(2) $\pi_k(W_i, (W_i)_0) \to \pi_k(W_{i-2}, (W_{i-2})_0)$ is zero for $2 \le i \le n$ and $0 \le k \le 5$.
(3) $\pi_k(W(i,j), W(i,j)_1) \to \pi_k(W(i,j+l), W(i,j+l)_1)$ is zero for $1 \le i < j \le n-l$ and $0 \le k \le 5$.
(4) $\pi_k(W_i, (W_i)_1) = 0$ for $1 \le i \le n$ and $0 \le k \le 5$.

To prove observation (1), note that $W(i,j)_0 = N(i,j)\times\{0\}$, let $t = 1-(1/i)$, and consider the following commutative diagram of inclusion-induced homomorphisms:

$$\begin{array}{ccc}
\pi_k(W(i,j), W(i,j)_0) & \longrightarrow & \pi_k(W(i-l,j), W(i-l,j)_0) \\
\zeta\downarrow & & \downarrow\zeta' \\
\pi_k(W(i,j), N(i,j)\times[0,t]) & \longrightarrow & \pi_k(W(i-l,j), N(i-l,j)\times[0,t]) \\
\eta\uparrow & & \uparrow\eta' \\
\pi_k(M(i,j)\times\{t\}, N(i,j)\times\{t\}) & \longrightarrow & \pi_k(M(i-l,j)\times\{t\}, N(i-l,j)\times\{t\}).
\end{array}$$

The vertical arrows are isomorphisms. To see that ζ is an isomorphism, consider the homotopy exact sequence of the triple

$$(W(i,j), N(i,j)\times[0,t], N(i,j)\times\{0\})$$

and use the fact that $\pi_*(N(i,j)\times[0,t], N(i,j)\times\{0\}) = 0$. Similarly, ζ' is an isomorphism. η and η' are isomorphisms because the inclusions which induce them are homotopy equivalences. Lemma 4.6 implies that the bottom horizontal arrow is zero. Hence the top horizontal arrow is zero.

To prove observation (2), note that $(W_i)_0 = N_{i+1}\times\{0\}$, let $t = 1-(1/i)$, and consider the following commutative diagram of inclusion-induced homomorphisms:

$$\begin{array}{ccc}
\pi_k(W_i, (W_i)_0) & \longrightarrow & \pi_k(W_{i+2}, (W_{i-2})_0) \\
\zeta\downarrow & & \downarrow\zeta' \\
\pi_k(W_i, N_{i+1}\times[0,t]) & \longrightarrow & \pi_k(W_{i-2}, N_{i-1}\times[0,t]) \\
\eta\uparrow & & \uparrow\eta' \\
\pi_k(V_i\times\{t\}, N_{i+1}\times\{t\}) & \longrightarrow & \pi_k(V_{i-2}\times\{t\}, N_{i-1}\times\{t\}).
\end{array}$$

The vertical arrows are isomorphisms for the reasons given in the proof of observation (1). Lemma 4.2 implies that the bottom horizontal arrow is zero. Thus, the top horizontal arrow is zero.

To prove observation (3), note that $W(i,j)_1 = V(i,j)\times\{1\}$, let $t=1-(1/j)$, and consider the following commutative diagram of inclusion-induced homorphisms:

$$\begin{array}{ccc} \pi_k(W(i,j),W(i,j)_1) & \longrightarrow & \pi_k(W(i,j+l),W(i,j+l)_1) \\ \zeta\downarrow & & \downarrow\zeta' \\ \pi_k(W(i,j),V(i,j)\times[t,1]) & \longrightarrow & \pi_k(W(i,j+l),V(i,j+l)\times[t,1]) \\ \eta\uparrow & & \uparrow\eta' \\ \pi_k(M(i,j)\times\{t\},V(i,j)\times\{t\}) & \longrightarrow & \pi_k(M(i,j+l)\times\{t\},V(i,j+l)\times\{t\}). \end{array}$$

The vertical arrows are isomorphisms for the reasons given in the proof of observation (1). Lemma 4.4 implies that the bottom horizontal arrow is zero. Thus, the top horizontal arrow is zero.

Observation (4) follows from the fact that the inclusion of $(W_i)_1 = V_i\times\{1\}$ into W_i is a homotopy equivalence.

Now let $t=0,1$ and $0\le k\le 5$. Suppose $\zeta\le\delta$ and $A\subset[\frac{1}{4},1]^{(\zeta)}$. We must find $B\subset[\frac{1}{4},1]^{(\zeta+\gamma)}$ such that $\operatorname{diam}B<\operatorname{diam}A+\gamma$ and $\pi_k(A^*,A^*\cap(M\times\{t\}))\to\pi_k(B^*,B^*\cap(M\times\{t\}))$ is zero. Since $\delta\le\frac{1}{8}$, $A\subset[\frac{1}{8},1]$. Since $n\ge 8(l+2)$, then there are integers $l+2\le i<j\le n$ such that $A\subset[i/n,j/n]$ and $(j-i)/n<\operatorname{diam}A+(2/n)$. Set $B=[(i-l-1)/n,\min\{(j+l+1)/n,1\}]$. Since $n>(l+2)/\gamma$ and $\frac{1}{4}-\zeta\le\inf A<(i+1)/n$, $\frac{1}{4}-(\zeta+\gamma)<\frac{1}{4}-\zeta-(l+2)/n<(i+l-1)/n$; so $B\subset[\frac{1}{4},1]^{(\zeta+\gamma)}$. Also, since $n\ge 2(l+2)/\gamma$, $\operatorname{diam}B\le(j-i)/n+(2l+2)/n<\operatorname{diam}A+(2l+4)/n\le\operatorname{diam}A+\gamma$. At this point we break the argument into four cases.

Case 1: $t=0$ and $j+l+1<n$. Since $A\subset((i-1)/n,(j+1)/n)$ and $B=[(i-l-1)/n,(j+l+1)/n]$,

$$A^*=p^{-1}(A)\subset W(i-1,j+1) \quad\text{and}\quad W(i-l-1,j+1)\subset p^{-1}(B)=B^*.$$

Observation (1) implies that

$$\pi_k(W(i-1,j+1),W(i-1,j+1)_0)\to\pi_k(W(i-l-1,j+1),W(i-l-1,j+1)_0)$$

is zero. Hence $\pi_k(A^*,A^*\cap(M\times\{0\}))\to\pi_k(B^*,B^*\cap(M\times\{0\}))$ is zero.

Case 2: $t=0$ and $n-l-1\le j\le n$. Since $A\subset((i-1)/n,1]$ and $B=[(i-l-1)/n,1]$, $A^*=p^{-1}(A)\subset W_{i-1}$ and $W_{i-3}\subset p^{-1}(B)=B^*$. Observation (2) implies that $\pi_k(W_{i-1},(W_{i-1})_0)\to\pi_k(W_{i-3},(W_{i-3})_0)$ is zero. Hence $\pi_k(A^*,A^*\cap(M\times\{0\}))\to\pi_k(B^*,B^*\cap(M\times\{0\}))$ is zero.

Case 3: $t=1$ and $j+l+1<n$. Since $A\subset((i-1)/n,(j+1)/n)$ and $B=[(i-l-1)/n,(j+l+1)/n]$,

$$A^*=p^{-1}(A)\subset W(i-1,j+1) \quad\text{and}\quad W(i-1,j+l+1)\subset p^{-1}(B)=B^*.$$

Observation (3) implies that

$\pi_k(W(i-1,j+1),W(i-1,j+1)_1)\to\pi_k(W(i-1,j+l+1),W(i-1,j+l+1)_1)$ is zero. Hence $\pi_k(A^*,A^*\cap(M\times\{1\}))\to\pi_k(B^*,B^*\cap(M\times\{1\}))$ is zero.

Case 4: $t=1$ and $n-l-1\le j\le n$. Since $A\subset((i-1)/n,1]$ and $B=[(i-l-1)/n,1]$, $A^*=p^{-1}(A)\subset W_{i-1}\subset p^{-1}(B)=B^*$. Observation (4) implies $\pi_k(W_{i-1},(W_{i-1})_1)=0$. Hence $\pi_k(A^*,A^*\cap(M\times\{1\}))\to\pi_k(B^*,B^*\cap(M\times\{1\}))$ is zero.

We have just completed the verification of condition (*) in the hypothesis of Lemma 5.1. Thus Lemma 5.1 implies that $f\colon W\to E$ is a δ-h-cobordism over $Y^{(\epsilon)}$. Hence, the hypotheses of Theorem 5.2 are now verified. Theorem 5.2 provides $f\colon W\to E$ with an ϵ-product structure over Y, where $f=\mathrm{id}_{M\times[0,1]}$ and $Y=[\frac{1}{2},1]$ where $\epsilon=\frac{1}{4}$. This means there is an embedding

$$e\colon M\times[0,1]\to M\times[0,1]$$

such that for every $x\in M$, $e(x,0)=(x,0)$, $\operatorname{diam}p(e(\{x\}\times[0,1]))<\frac{1}{4}$, and $p^{-1}([\frac{1}{2},1])\subset e(M\times[0,1])$.

Set $i=(\frac{3}{4})n+1$ and $j=n/2$. Then i and j are integers. We will prove that

$$V_n\times\{1\}\subset e(N_i\times\{1\})\subset V_j\times\{1\}.$$

$(M\times\{1\})\cap e(M\times[0,1])\subset e(M\times\{1\})$ because $e(M\times\{0\})=M\times\{0\}$ and $e(M\times(0,1))=\operatorname{int}e(M\times[0,1])\subset\operatorname{int}M\times[0,1]=M\times(0,1)$. It follows that since $V_n\times\{1\}\subset W_n\subset p^{-1}(\{1\})\subset e(M\times[0,1])$, $V_n\times\{1\}\subset e(M\times\{1\})$. Suppose $x\in M$ and $e(x,1)\in V_n\times\{1\}$. Then $1-p(x,0)=p(e(x,1))-p(e(x,0))<\frac{1}{4}$. So $p(x,0)\in(\frac{3}{4},1]$. Thus, $(x,0)\in W_{i-1}\cap(M\times\{0\})=N_i\times\{0\}$. Hence, $x\in N_i$. This proves $V_n\times\{1\}\subset e(N_i\times\{1\})$. Now let $y\in N_i$. Then $p(e(y,0))=p(y,0)\in p(N_i\times\{0\})\subset p(W_{i-1})\subset[\frac{3}{4},1]$. So

$$(\tfrac{3}{4})-p(e(y,1))\le p(e(y,0))-p(e(y,1))<\tfrac{1}{4}.$$

Hence, $p(e(y,1))\in(\frac{1}{2},1]$. Therefore, $e(y,1)\in W_j$. Since $W_j\subset p^{-1}([\frac{1}{2},1])\subset e(M\times[0,1])$, $\operatorname{int}W_j\subset e(M\times(0,1))$. Consequently, $e(y,1)\in\partial W_j$. Clearly $\partial W_j\subset(N_{j+1}\times\{0\})\cup(V_j\times\{1\})\cup\operatorname{cl}((M\times[0,1])-W_j)$. $e(y,1)\notin N_{j+1}\times\{0\}$ because $N_{j+1}\times\{0\}\subset e(M\times\{0\})$. $e(y,1)\notin\operatorname{cl}((M\times[0,1])-W_j)$ because $p(\operatorname{cl}((M\times[0,1])-W_j))\subset[0,\frac{1}{2}]$ and $p(e(y,1))>\frac{1}{2}$. We conclude that $e(y,1)\in V_j\times\{1\}$. This proves $e(N_i\times\{1\})\subset V_j\times\{1\}$.

The projection of $M\times\{1\}$ to M carries $e(N_i\times\{1\})$ to a set N satisfying $V_n\subset N\subset V_j$. N is homeomorphic to $S^1\times B^3$ because $N\cong N_i$ and N_i is a regular neighborhood of a 1-sphere in the orientable manifold M. $X\subset\operatorname{int}N\subset N\subset U$ because $X\subset\operatorname{int}V_n$ and $V_j\subset V_0=U_0\subset U$. □

REMARK. It is only at the end of the proof of Theorem 1, when the controlled h-cobordism theorem is invoked, that the hypothesis that X have the shape of an S^1-like continuum is really essential. It is possible to find versions of the neighborhood constructions in Section 3 and the homotopy lemmas in Section 4 which work for any X having fundamental dimension 1. In that case, the neighborhoods would have spines that are arbitrary compact

1-dimensional polyhedra, and thus their fundamental groups would be finitely generated free groups. But the only such group for which the controlled h-cobordism theorems of [3] are known to hold is the free group on one generator, **Z**.

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F. D. Ancel
Department of Mathematics
University of Wisconsin - Milwaukee
Milwaukee, WI 53201

V. T. Liem
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350

Gerard A. Venema
Department of Mathematics
Calvin College
Grand Rapids, MI 49546